Exercise 39

(a) If $f(x) = (x^2 - 1)/(x^2 + 1)$, find $f'(x)$ and $f''(x)$.

(b) Check to see that your answers to part (a) are reasonable by comparing the graphs of f, f', and f''.

Solution

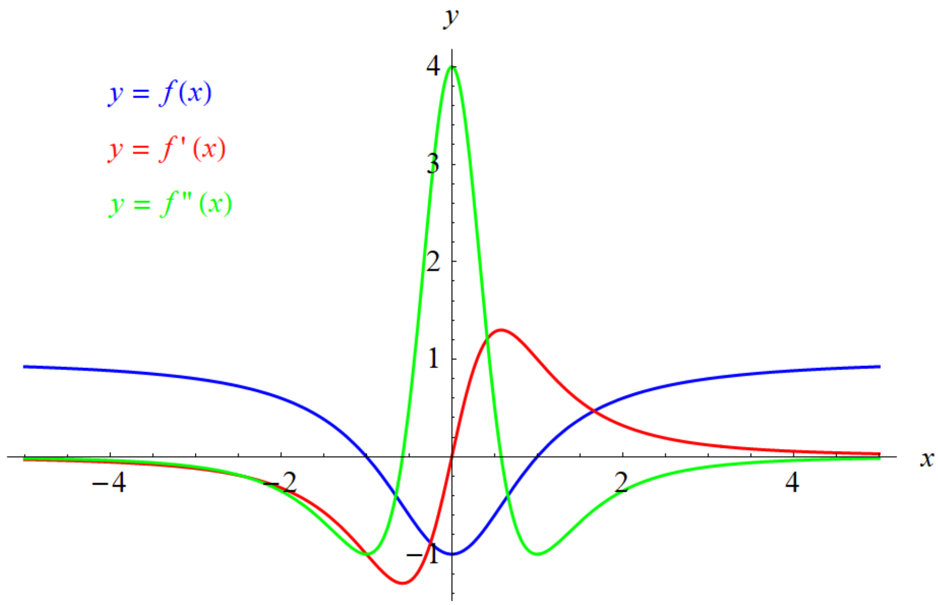
Evaluate the derivative using the quotient rule.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right) \\ &= \frac{\left[\frac{d}{dx}(x^2-1)\right](x^2+1) - \left[\frac{d}{dx}(x^2+1)\right](x^2-1)}{(x^2+1)^2} \\ &= \frac{(2x)(x^2+1) - (2x)(x^2-1)}{(x^2+1)^2} \\ &= \frac{4x}{(x^2+1)^2} \end{aligned}$$

Evaluate the second derivative using the quotient rule again.

$$\begin{aligned} f''(x) &= \frac{d}{dx}[f'(x)] \\ &= \frac{d}{dx}\left(\frac{4x}{(x^2+1)^2}\right) \\ &= \frac{\left[\frac{d}{dx}(4x)\right](x^2+1)^2 - \left\{\frac{d}{dx}[(x^2+1)^2]\right\}(4x)}{(x^2+1)^4} \\ &= \frac{(4)(x^2+1)^2 - \left\{\left[\frac{d}{dx}(x^2+1)\right](x^2+1) + (x^2+1)\left[\frac{d}{dx}(x^2+1)\right]\right\}(4x)}{(x^2+1)^4} \\ &= \frac{4(x^2+1)^2 - \left[(2x)(x^2+1) + (x^2+1)(2x)\right](4x)}{(x^2+1)^4} \\ &= \frac{4 - 8x^2 - 12x^4}{(x^2+1)^4} \\ &= \frac{-4(x^2+1)(3x^2-1)}{(x^2+1)^4} \\ &= -\frac{4(3x^2-1)}{(x^2+1)^3} \end{aligned}$$

Below is a graph of the function and its derivatives versus x.



$f'(x)$ is positive wherever $f(x)$ increases, $f'(x)$ is zero wherever the slope of $f(x)$ is zero, and $f'(x)$ is negative wherever $f(x)$ is decreasing.

Similarly, $f''(x)$ is positive wherever $f'(x)$ increases, $f''(x)$ is zero wherever the slope of $f'(x)$ is zero, and $f''(x)$ is negative wherever $f'(x)$ is decreasing. The answers in part (a) are reasonable then.